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Journal of Pure and Applied Algebra 12 (1978) 147–151.
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# $p$-ADIC GROUP RINGS WITH NILPOTENT UNIT GROUPS

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Communicated by P. Hilton
Received 7 March 1977

## 1. Introduction

Let $RG$ be the group ring of a group $G$ over a commutative ring $R$ with unity and $U(RG)$ its unit group. In the case where $G$ is finite and $R$ is either $\mathbf{Z}$ the ring of rational integers or $\mathbf{Z}_p$ the ring of $p$-adic integers, necessary and sufficient conditions for $U(RG)$ to be nilpotent were given in [3].

For infinite groups, S.K. Sehgal and H.J. Zassenhaus [5] proved the following:

**Theorem.** *$U(\mathbf{Z}G)$ is nilpotent if and only if $G$ is nilpotent and the torsion subgroup $T$ of $G$ satisfies one of the following:*

(i) *$T$ is central in $G$.*

(ii) *$T$ is an abelian 2-group and for $x \in G$, $t \in T$*

$$xtx^{-1} = t^{\delta(x)}, \qquad \delta(x) = \pm 1.$$

(iii) *$T = E \times \mathcal{Q}$ where $E$ is an elementary abelian 2-group, and $\mathcal{Q}$ is the quaternion group of order 8. Moreover, $E$ is central in $G$ and conjugation by $x \in G$ induces on $\mathcal{Q}$ one of the four inner automorphisms.*

In this paper we prove the following result:

**Theorem.** *Let $T$ be the set of torsion elements of a group $G$ and set $R = \mathbf{Z}_{(p)}$ a localization of $\mathbf{Z}$ at a prime ideal $(p)$ or $R = \mathbf{Z}_p$. Then $U(RG)$ is nilpotent if and only if $G$ is nilpotent and one of the following conditions holds:*

(i) *$T$ is central in $G$*

(ii) *$p = 2$, $T$ is an abelian 2-group and for all $x \in G$, $t \in T$*

$$xtx^{-1} = t^{\delta(x)}, \qquad \delta(x) = \pm 1.$$

In what follows, $\mathbf{Z}_p$ will denote the field of integers modulo $p$ and for elements $x_1, \ldots, x_n$ in a group $X$ we set:

*This work was written while the author was on leave at the Memorial University of Newfoundland on an exchange program by NRC (Canada) and CNPq (Brasil).

$$(x_1, x_2) = x_1^{-1} x_2^{-1} x_1 x_2,$$

$$(x_1, \ldots, x_n) = ((x_1, \ldots, x_{n-1}), x_n).$$

Also, $X^{(n)}$ will denote the subgroup of $X$ generated by the set

$$\{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in X\}.$$

The author is grateful to Dr. M.M. Parmenter for helpful conversations while this work was done.

## 2. Preliminary results

**Proposition 2.1.** *Let $\mathcal{Q} = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ bab^3 = a^3 \rangle$ denote the quaternion group of order* 8. *Then $U(\mathbf{Z}_{(p)}G)$ is not nilpotent.*

**Proof.** Let $\mathbf{H}_p$ denote the following subring of the ring of rational quaternions:

$$\mathbf{H}_p = \{\alpha_0 + \alpha_1 \mathrm{i} + \alpha_2 \mathrm{j} + \alpha_3 \mathrm{k} \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{Z}_{(p)}\}.$$

We shall first show that $U(\mathbf{H}_p)$ is not nilpotent.

For an element of the form $\alpha = m/d + (n/d)\mathrm{i} \in \mathbf{H}_p$ we shall say that $\alpha$ has property (p) if $p \nmid m$ and $p \mid n$.

If an element $\alpha$ has property (p), then $(j, \alpha)$ has also property (p) since:

$$(j, \alpha) = \frac{m^2 - n^2}{m^2 + n^2} + \frac{2mn}{m^2 + n^2}\mathrm{i}$$

and, for such an element $\alpha$ set: $\alpha_0 = \alpha$, $\alpha_k = (j, \alpha_{k-1})$.

Then this is a sequence of commutators such that $\alpha_k \neq 1$ for all $k$; thus $U(\mathbf{H}_p)$ is not nilpotent.

Now, we define a $\mathbf{Z}_{(p)}$-linear function $\omega : \mathbf{Z}_{(p)}\mathcal{Q} \to \mathbf{H}_p$ by:

$$\omega(1) = 1, \quad \omega(a) = \mathrm{i}, \quad \omega(b) = \mathrm{j}, \quad \omega(ab) = \mathrm{k},$$

$$\omega(a^2) = -1, \quad \omega(a^3) = -\mathrm{i}, \quad \omega(a^2 b) = -\mathrm{j}, \quad \omega(a^3 b) = -\mathrm{k}.$$

It is easy to see that $\omega$ is a ring homomorphism and if $\alpha_0 = x + ya$; $x, y \in \mathbf{Z}$ is any element with property (p) then $\alpha_0$ is a unit in $\mathbf{Z}_{(p)}\mathcal{Q}$ whose inverse is

$$\alpha_0^{-1} = \frac{-x^3}{y^4 - x^4} + \frac{yx^2}{y^4 - x^4} a - \frac{xy^2}{y^4 - x^4} a^2 + \frac{y^3}{y^4 - x^4} a^3$$

and $\omega(\alpha_0) = \alpha \in \mathbf{H}_p$ has property (p).

Thus, we can exhibit a sequence of commutators in $U(\mathbf{Z}_{(p)}\mathcal{Q})$ whose image is never 1; consequently $U(\mathbf{Z}_{(p)}\mathcal{Q})$ is not nilpotent.

Now, let $F$ be a ring, $\alpha$ an automorphism of $F$ and $x \in G$. We shall consider the skew group ring $F^\alpha\langle x \rangle = \{\sum_i f_i x^i \mid f_i \in F\}$ with addition and equality defined componentwise and $xf = f^\alpha x$ for all $f \in F$.

**Lemma 2.2.** *Let $\alpha$ be an automorphism of finite order $s$ on $F = \mathbf{Z}_{(p)}[\xi]$ where $\xi$ is a primitive root of unity and suppose that in $F^{\alpha}\langle x\rangle$ we have:*

$$(f, \underbrace{x, \ldots, x}_{m}) = 1 \quad \text{for all } f \in U(F).$$

*Then, $F^{\alpha}\langle x\rangle = F\langle\alpha\rangle$, i.e. $\alpha$ is the identity automorphism.*

**Proof.** As in [2, Proposition 2.3] it follows that the automorphism group $\{I, \alpha, \ldots, \alpha^{s-1}\}$ satisfies an equation:

$$h(f, \alpha(f), \ldots, \alpha^{s-1}(f)) = 0 \quad \text{for all } f \in U(F)$$

where $h(X_0, X_1, \ldots, X_{s-1})$ in a non trivial polynomial.

Since $\alpha : \mathbf{Z}_{(p)}[\xi] \to \mathbf{Z}_{(p)}[\xi]$ is such that $\alpha \mid \mathbf{Z}_{(p)}[\xi] =$ identity and $\alpha(\xi) = \xi^{i}$ for some positive integer $i$, it can be extended in a natural way to an automorphism $\alpha_* : \mathbf{Q}(\xi) \to \mathbf{Q}(\xi)$ and we then have:

$$h(x, \alpha_*(x), \ldots, \alpha_*^{s-1}(x)) = 0 \quad \text{for all } x \in U(\mathbf{Q}(\xi)).$$

Again, as in [2, Proposition 2.3] we can conclude that $\alpha$ is the identity automorphism.

**Proposition 2.3.** *Let $G$ be a group such that $\mathbf{Z}_{(p)}G$ is nilpotent for some prime rational integer $p \neq 2$ and such that its torsion subgroup $T$ verifies condition* (ii) *of the theorem. Then $T$ is central.*

**Proof.** Let $T_0$ be a finite subgroup of $T$ and $x \in G \setminus T$. Since $T_0$ is himself an abelian group, we know that:

$$\mathbf{Q}T_0 \cong \bigoplus_i \mathbf{Q}(\xi_i),$$

where $\xi_i$ is a primitive $d_i$-root of unity for some divisor $d_i$ of $|T_0|$.

This actually means that we can find a set of primitive central idempotents $\{e_1, \ldots, e_t\}$ such that $\mathbf{Q}T_0 = \bigoplus_i (\mathbf{Q}T_0)e_i$ and $(\mathbf{Q}T_0)e_i \cong \mathbf{Q}(\xi_i)$, $1 \leq i \leq t$.

We claim that the elements $e_i$, $1 \leq i \leq t$, actually belong to $\mathbf{Z}_{(p)}T_0$. In fact, assume that $e \in \mathbf{Q}G$ is an idempotent such that $e = (1/m)\Sigma_g x(g)g$ where $p \mid m$ and $\Sigma_g x(g)g \in \mathbf{Z}T_0 \setminus p\mathbf{Z}T_0$.

Then $\alpha = me$ is an element in $\mathbf{Z}T_0$ verifying $\alpha^2 = m\alpha$. If $\alpha_1$ is the image of $\alpha$ in the homomorphism $\mathbf{Z}T_0 \to \mathbf{Z}_pT_0$ induced by the natural homomorphism $\mathbf{Z} \to \mathbf{Z}_p$, then $\alpha_1^2 = 0$.

Since $T_0$ is abelian and $p \nmid |T_0|$, $\mathbf{Z}_pG$ is a direct sum of fields. So it contains no nilpotent elements and $\alpha_1 = 0$. Hence $\alpha \in p\mathbf{Z}T_0$, a contradiction.

Thus, we can write:

$$\mathbf{Z}_{(p)}T_0 = \bigoplus_i (\mathbf{Z}_{(p)}T_0)e_i \tag{2.1}$$

Now, we claim that if $(\mathbf{Q}T_0)e \cong \mathbf{Q}(\xi)$ for some primitive $d$-root of unity $\xi$, then $(\mathbf{Z}_{(p)}T_0)e \cong \mathbf{Z}_{(p)}[\xi]$.

In fact, if $\Phi : (\mathbf{Q}T_0)e \to \mathbf{Q}(\xi)$ is the given isomorphism, it is clear that $\mathbf{Z}_{(p)} \subset \Phi((\mathbf{Z}_{(p)}T_0)e)$. Also, if $\gamma \in (\mathbf{Q}T_0)e$ is such that $\Phi(\gamma) = \xi$ then $\gamma^d = e$; thus, $\gamma \in \mathbf{Z}_{(p)}T_0$ and $\mathbf{Z}_{(p)}[\xi] \subset \Phi((\mathbf{Z}_{(p)}T_0)e)$.

To prove the opposite inclusion, it will suffice to show that if $g \in T_0$ then $\Phi(g) \in \mathbf{Z}_{(p)}[\xi]$. Suppose $\Phi(g) = (1/p^m)f(\xi)$ with $m \geq 1$, $f(x) \in \mathbf{Z}_{(p)}[X]$; then

$$\Phi\left(\frac{1}{p^m} f(\gamma)\right) = \Phi(g) \quad \text{and} \quad g^{-1} \cdot \frac{1}{p^m} f(\gamma) = e \in \mathbf{Z}_{(p)}T_0,$$

a contradiction.

Hence, (2.1) now yields:

$$\mathbf{Z}_{(p)}T_0 \cong \bigoplus_i \mathbf{Z}_{(p)}[\xi]. \tag{2.2}$$

Finally, let $\alpha$ be the automorphism of $\mathbf{Z}_{(p)}T_0$ induced by conjugation by $x$. It follows from [4, Theorem 4.8] and [4, 2.5] that every idempotent is central, thus:

$$(\mathbf{Z}_{(p)}T_0)^\alpha\langle x\rangle \cong \left(\bigoplus_i \mathbf{Z}_{(p)}[\xi_i]\right)^\alpha \langle x\rangle \cong \bigoplus_i (\mathbf{Z}_p[\xi_i])^\alpha\langle x\rangle. \tag{2.3}$$

Since the unit group of each $(\mathbf{Z}_p[\xi_i])^\alpha\langle x\rangle$ is nilpotent, the previous lemma shows that $\alpha$ is the identity. This implies that $T$ is central.

## 3. Proof of the theorem

Since the nilpotency of $U(RG)$ implies in both cases that $U(\mathbf{Z}G)$ is also nilpotent, the necessity of the conditions follows immediately from the theorem by Sehgal and Zassenhaus stated in the introduction and Propositions 2.1 and 2.3. Thus it only remains to prove sufficiency.

First, we note that it follows from the theorem in [2] that if $T$ is central in $G$ then $U(\mathbf{Q}_pG)$ is nilpotent, where $\mathbf{Q}_p$ denotes the $p$-adic completion of the field of rational numbers (also note that the restriction $p \neq 2, 3$ in that theorem does not apply in the proof of (ii) $\Longrightarrow$ (iii) which is needed here). Since $U(RG) \subset U(\mathbf{Q}_pG)$ in both cases, it follows that condition (i) in our theorem is sufficient.

Now assume that (ii) holds. Then the proof of [5, Lemma 3.6] is still valid, except perhaps for the use of [5, Lemma 3.1]. But it follows from [1, Lemma 2.4] that $\mathbf{Z}_2G$ contains no non-trivial idempotents. Also, $\mathbf{Z}_2G \subset \mathbf{Q}_2G$ and the latter contains no non-zero nilpotent elements, as follows from the proof of [4, proposition 4.1]. Thus, the lemma holds.

Hence, we obtain:

$$U(\mathbf{Z}_2G) = U(\mathbf{Z}_2T) \cdot G,$$

Now, we claim that:

$$U(\mathbf{Z}_2G)^{(n)} \subset G^{(n)}U(\mathbf{Z}_2T). \tag{3.1}$$

In fact, first we notice that if $x = \sum k(t)t \in U(\mathbf{Z}_2T)$, $a \in G$ then $x^a = a^{-1}xa = \sum k(t)t^{\delta(a)} \in U(\mathbf{Z}_2T)$ thus:

$$a^{-1}U(\mathbf{Z}_2T)a \subset U(\mathbf{Z}_2T) \quad \text{for all } a \in G. \tag{3.2}$$

Given $\alpha, \beta \in U(\mathbf{Z}_2T)$ we can write them on the form $\alpha = u^{-1}g^{-1}$, $\beta = hv$ with $g, h \in G$ and $u, v \in U(\mathbf{Z}_2T)$. Then:

$$\beta^{-1}\alpha^{-1}\beta = (g^h)^v(u^h)^v. \tag{3.3}$$

From (3.2) it follows immediately that $(u^h)^v \in U(\mathbf{Z}_2T)$.

On the other hand: $(g^h)^v = vg^hv^{-1} = g^h\bar{v}v^{-1}$, where $\bar{v}$ is such that $(g^h)^{-1}v(g^h) = \bar{v}$.

Hence, in (3.3) we get:

$$\beta^{-1}\alpha^{-1}\beta = g^h \cdot w, \quad \text{with } w \in U(\mathbf{Z}_2T). \tag{3.4}$$

Finally:

$$\beta^{-1}\alpha^{-1}\beta\alpha = g^hwu^{-1}g^{-1} = g^hg^{-1}\bar{w}, \quad \text{with } \bar{w} \in U(\mathbf{Z}_2T),$$

consequently:

$$(\beta, \alpha) = (g, h) \quad w \in G^{(1)}U(\mathbf{Z}_2T).$$

An induction argument now shows that (3.1) holds.

Since $G$ is nilpotent and $U(\mathbf{Z}_2T)$ is abelian it follows that $U(\mathbf{Z}_2G)$ is nilpotent, hence $U(\mathbf{Z}_{(2)}G)$ is also nilpotent and the proof is complete.

As a final remark we would like to notice that the proofs above actually show that the theorem holds for any integral domain of characteristic 0 where all the primes except one are invertible; for example, the ring of a non-archimedean valuation on an algebraic number field.

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